PERTH MODERN SCHOOL
Exceptional schooling. Exceptional students.
Independent Public School

Course Mathematics Specialist Year 11

Student name: ____________________ Teacher name: _______________

Date: Monday 23rd March

Task type: **Investigation**

Time allowed for this task: **45 mins**

Number of questions: **3**

Materials required: Calculator with CAS capability (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE Examinations.

Marks available: **35 marks**

Task weighting: **10%**

Formula sheet provided: **Yes**

Note: All part questions worth more than 2 marks require working to obtain full marks.

Robots O, X and Y are playing a game using a row of 9 boxes:

The three robots take turns to randomly add their symbol to an empty box, until the grid is complete, e.g.

O	Y	Y	X	O	O	Y	X	X

They use a scoring system where each row of 3 identical symbols in a completed grid scores a point for that robot.

1. [9 = 2+2+2+3 marks]

   a) Briefly explain why the total number of distinct completed grids is $\binom{9}{3} \times \binom{6}{3} \times \binom{3}{3}$, and evaluate this expression.

For each of the $\binom{9}{3}$ choices for the positions of the 3 Xs, there are $\binom{6}{3}$ choices for the positions of the 3 Ys, and then $\binom{3}{3}$ for the 3 Os.

✓ refers to counting choices for positions

$$\binom{9}{3} \times \binom{6}{3} \times \binom{3}{3} = 1680$$ ✓ correct value.

   b) Calculate each of the following probabilities. Do not simplify your answers.

      i. Robots X, Y and O each score 1 point after a game.

Xs, Ys and Os must be together, e.g. XXXYYYOOO

Total number of possibilities $= 3!$ ✓ 3! possibilities
$= 6$

$P(\text{all 3 score 1 point}) = \frac{6}{1680}$ ✓ correct probability

Question 1(b) continued

ii. Robot Y scores 1 point (and the other robots may or may not score points).

There are 7 positions for YYY, and for each of those there are $\binom{6}{3}$ arrangements of the remaining symbols.

$\therefore$ N° of possibilities $= 7 \times \binom{6}{3}$ ✓ calculates N° of possibilities as $7 \times \binom{6}{3}$

$\therefore$ $P(Y \text{ scores } 1) = \frac{140}{1680}$ ✓ correct probability

iii. Robots X and Y each score 1 point, but Robot O scores 0 points after a game.
[Hint: begin by listing all arrangements without OOO, and where XXX comes before YYY.]

Arrangements with X before Y:

$$\left.\begin{array}{l} XXXOYYYOO \\ XXXOOYYYO \\ OXXXYYYOO \\ OXXXOYYYO \\ OXXXOOYYY \\ OOXXXYYYO \\ OOXXXOYYY \end{array}\right\} 7 \text{ arrangements.}$$

✓ determines 7 arrangements with X before Y

Also 7 arrangements with Y before X.

$\therefore$ N° of possibilities $= 2 \times 7$

✓ multiplies by 2

$\therefore$ $P(X, Y \text{ score } 1, O \text{ scores } 0) = \frac{14}{1680}$ ✓ correct probability

Accept other methods with sufficient working.

The three robots now decide to play using a 3x3 grid. Once again, they take turns to add their own symbol **randomly** to one of the empty spaces on the grid, until a completed grid is obtained. They use all nine squares, so that a completed grid always contains 3 of each symbol, e.g.

O	X	Y
X	O	Y
Y	X	O

In a completed grid, each row of 3 identical symbols (horizontal, vertical or diagonal) scores a point for that player. E.g. in the grid above, Robot O scores 1 point and Robots X and Y score 0 points.

2. [16 = 2+2+2+2+2+3+3 marks]

a) Calculate the total number of distinct completed grids.

Total N° of grids $= \binom{9}{3} \times \binom{6}{3} \times \binom{3}{3}$ ✓ calculation (accept $\binom{9}{3}\times\binom{6}{3}$)

$= 1680$ ✓ correct number

b) Calculate each of the following probabilities. Do not simplify your answers.

i. Robots X, Y and O each score 1 point after a game.

Possibilities have either 3 vertical rows, e.g.

X	Y	O
X	Y	O
X	Y	O

or 3 horizontal rows, e.g.

X	X	X
Y	Y	Y
O	O	O

There are 3! of each type.

✓ reasons $2 \times 3!$ possibilities

$\therefore P(X, Y, O \text{ each score } 1) = \dfrac{12}{1680}$ ✓ correct probability

ii. Robots X and Y each score 1 point, but Robot O scores 0 points after a game.

X	Y	O
X	Y	O
X	Y	O

Impossible! ✓ reasons impossible

$P(X, Y \text{ each score } 1, O \text{ scores } 0) = 0$ ✓ correct probability

iii. Robot Y scores 1 point (and the other robots may or may not score points).

8 positions for 3 Ys (h, v, diag) ✓ identifies 8 positions for 3 Ys.

For each of these there are $\binom{6}{3}$ arrangements of remaining letters. $\therefore$ N° of possibilities $= 8 \times \binom{6}{3}$

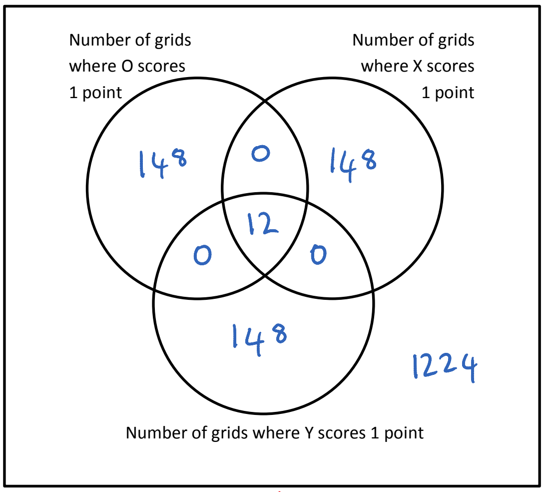
$P(Y \text{ scores } 1) = \dfrac{160}{1680}$ ✓ correct probability

c) Given your answer to part b (ii), what is the probability that Robots Y and O each score 1 point after a game, but Robot X scores 0? Explain your answer.

Roles of all 3 robots are equivalent,
So same reasoning applies. ✓ correct explanation

Hence P(Y, O score 1 and X scores 0) = 0 ✓ correct probability

d) Calculate the probability that Robot Y scores 1 point after a game, but Robots X and O both score 0 points.
[Hint: the Venn diagram below may be useful, together with your answers to previous parts of the question.]



✓ uses Y circle of Venn diagram
(not required for full marks)

N° of grids where Y scores 1 point = 160. (part b iii)

∴ N° of grids where Y scores 1 and X, O score 0

= 160 − 12 ✓ subtracts 12 from 160

= 148

∴ $P(Y \text{ scores } 1, X, O \text{ score } 0) = \frac{148}{1680}$ ✓ correct probability

e) Calculate the probability that Robots O, X and Y each score 0 points after a game.
[Hint: use the Venn diagram again.]

Using Venn diagram: ✓ completes Venn diagram above (again, not required)

N° of grids where X, Y, O each score 0 is:

1680 − 3 × 148 − 12 = 1224 ✓ subtracts to determine number scoring 0

∴ $P(X, Y, O \text{ each score } 0) = \frac{1224}{1680}$ ✓ correct probability

Robot Y now has to depart, leaving Robots X and O to play together once again.

They decide to play a new game using a 4x4 grid. The rules are exactly the same as before (they take turns to randomly add their symbol), except that now each row of **4** identical symbols (horizontal, vertical or diagonal) in a completed grid scores 1 point for that player (a row of only 3 identical symbols does not score any points).

E.g.

X	O	O	O
X	X	O	X
X	O	X	O
X	O	O	X

scores a total of 2 points for Robot X (because there is both a vertical and a diagonal row of 4 Xs) and 0 points for Robot O.

3. [10 = 2 + 3 + 3 + 2 marks]

a) Calculate the total number of possible completed grids, showing working.

N° of grids $= \binom{16}{8} \times \binom{8}{8}$ ✓ calculation (accept $\binom{16}{8}$)

$= 12870$ ✓ correct number

b) Calculate the probability that Robot X and Robot O each score exactly 2 points after a game. Do not simplify your answer.

X	X	O	O
X	X	O	O
X	X	O	O
X	X	O	O

N° of grids with 2 vertical rows of each symbol

$= \binom{4}{2}$

$= 6$

N° of grids with 2 horizontal rows of each symbol

$= \binom{4}{2}$

$= 6$

✓ identifies vertical and horizontal cases

✓ calculates $\binom{4}{2}$ possibilities in each case

$\therefore P(2 \text{ points each}) = \frac{12}{12870}$

✓ correct probability

c) Calculate the probability that Robot X scores 2 points and Robot O scores 0 points after a game. Do not simplify your answer.

Possible grids:

1 v row of Xs + 1 h row of Xs : $4 \times 4 \times \binom{9}{1}$

1 v row of Xs + 1 diag row of Xs : $2 \times 4 \times \binom{9}{1}$

1 h " " " " " " " : $2 \times 4 \times \binom{9}{1}$

2 diag. rows of Xs : 1

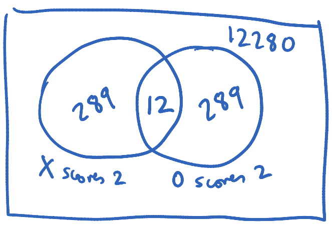
✓ identifies 4 cases for 2 rows of 4 Xs

✓ multiplies by $\binom{9}{1}$ in 3 of those cases

Total N° of grids $= 9(16+8+8)+1$
$= 289$

$\therefore$ $P(X \text{ scores } 2 \text{ and } O \text{ scores } 0) = \dfrac{289}{12870}$ ✓ correct probability

d) Calculate the probability that both robots score at most 1 point after a game.
[Hint: use a Venn diagram.]



✓ correct use of Venn diagram
OR alternative method

N° of games in which both robots score at most 1 point $= 12870 - 2 \times 289 - 12$
$= 12280$

$P(\text{both score} \leq 1 \text{ point}) = \dfrac{12280}{12870}$ ✓ correct probabity